Fall 2013 **Math 601** Name: ______________________

Sample Quiz 9

Question 1. (10 pts)

(a) Determine whether the function $f(z) = |z|^2$ is analytic on $\mathbb{C}$.

> **Solution:** $f(z) = x^2+y^2$. So the real part is $u(x,y) = x^2+y^2$ and the imaginary part is $v(x,y) = 0$. We have
>
> $$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y$$
>
> $$\frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0.$$
>
> The Cauchy-Riemann equations are not satisfied. Therefore, f is not analytic on $\mathbb{C}$.

(b) Determine whether the function $g(z) = \cos(x) + i\sin(y)$ is analytic on $\mathbb{C}$, where $z = x + iy$.

> **Solution:** This is similar to part (a). Try to test whether Cauchy-Riemann equations are satisfied.

Question 2. (10 pts)

Evaluate the following integrals.

(a)

$$\int_C \frac{z}{(z-2)^2} dz$$

where C is the unit circle $\{z \in \mathbb{C} : |z| = 1\}$ oriented counterclockwise.

Solution: Notice that the function $f(z) = \frac{z}{(z-2)^2}$ is analytic on the region $\Omega = \mathbb{C}\backslash\{2\}$. The unit circle $C = \{z \in \mathbb{C} : |z| = 1\}$ is contained in Ω with the inside of C lying in Ω. So we can apply Cauchy's Theorem, which implies that

$$\int_C \frac{z}{(z-2)^2} dz = 0$$

(b)

$$\int_C z^3 dz$$

where C is the upper semicircle of radius 1 starting at 1 and ending at -1.

Solution: Note that $F(z) = \frac{z^4}{4}$ is the antiderivative of z^3. Since the domain of $f(z) = z^3$ is $\mathbb{C}$, hence simply-connected, and the curve C is clearly contained in $\mathbb{C}$. We know that

$$\int_C z^3 dz = F(-1) - F(1) = 0$$